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Critical Slowing Down in One-Dimensional Maps and Beyond

Bailin Hao[1,2]

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This is a brief review on critical slowing down near the Feigenbaum period-doubling bifurcation points and its consequences. The slowing down of numerical convergence leads to an "operational" fractal dimension $D = 2/3$ at a finite order bifurcation point. There is a cross-over to $D_0 = 0.538\ldots$ when the order goes to infinity, i.e., to the Feigenbaum accumulation point. The problem of whether there exists a "super-scaling" for the dimension spectrum D_q^W that does not depend on the primitive word W underlying the period-n-tupling sequence seems to remain open.

KEY WORDS: Period doubling attractor; fractal dimension; critical slowing down.

1. INTRODUCTION

Feigenbaum's renormalization group derivation of the two universal exponents δ and $\alpha^{(1,2)}$ is more significant than the discovery of the two numerical constants themselves. Pedagogically speaking, it provides a nice paradigm to elucidate the idea and technique of renormalization group approach in statistical physics without requiring the heavy machinery of quantum field theory or sound knowledge of phase transitions and critical phenomena. Almost everything can be shown by using rather elementary mathematics. This is true not only in "equilibrium" properties (see, e.g., the review ref. 3), but also in phenomena related to "critical dynamics". Since the latter has been studied mainly by Chinese authors and the results might not be widely known I take the opportunity of

[1]T-Life Research Center, Fudan University, Shanghai 200433, China.
[2]Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China; e-mail: hao@itp.ac.cn

celebrating the seminal papers of Feigenbaum to provide a brief review of this development.

2. FRACTALS IN MAPS OF THE INTERVAL

Let us start with the question: Where can one encounter objects with a genuine fractal dimension in the limiting sets of one-dimensional mappings? A one-dimensional map has only one Lyapunov exponent λ. The Kaplan–Yorke conjecture[4] on the relation between the Lyapunov exponents and Lyapunov dimension implies that when $\lambda > 0$ the dimension is 1, when $\lambda < 0$ the dimension is 0. The only possibility to have non-integer dimension occurs at parameter values where the Lyapunov exponent itself is zero.

There are three cases of isolated points on the parameter axis where the Lyapunov exponent is exactly zero:

1. At a period-doubling bifurcation point, e.g., that from a fixed point to a 2-cycle. The fact that the Lyapunov exponent has a negative value on both sides of the bifurcation implies a zero dimension on both sides. Continuity suggests that the dimension should be $D = 0$ right at the bifurcation point as it actually is.

2. At the Feigenbaum limit of the infinite period-doubling cascade the trajectory may have a self-similar structure with a fractal dimension. Indeed, it was calculated by Grassberger[5] in 1981 to be $D_0 = 0.538\ldots$.

3. At a tangent bifurcation where the Lyapunov exponent drops from a positive value in the intermittently chaotic regime to a negative value in the periodic regime it must go through zero. There might be a fractal whose dimension should be easily calculable within the mean field framework of the theory of intermittency. In fact, it is $D = 1/2$.

That is all and the story might end at this point. However, this is just the beginning of my story on Feigenbaum.

3. SLOWING DOWN EXPONENT NEAR BIFURCATION POINT

In the summer of 1981 I was visiting Brussels and was attracted by the upsurge of interest in chaotic dynamics during one of the Haken's synergetics meeting at Schloss Elmau.[6] I started to do numerical work on differential equations in order to get some feeling on chaos. This study resulted in discovering a period-doubling cascade up to the 13th ($2^{13} = 8192$) bifurcation in the periodically forced Brusselator[7] and in eventually developing the symbolic dynamics analysis of ODEs many years later.[8]

The new Cyber computer at the Free University of Brussels was quite old-fashioned by present-day standard. One had to indicate on a punched card the CPU time needed for the job. If one put too long a time the operating system would schedule the job for the weekend and the result could not come soon. If one put too short a time, the job might terminate without yielding any result. The convergence of the calculation got slower and slower when the parameter was approaching a bifurcation point.

With my experience of working on critical dynamics near phase transitions[9] I immediately understood it as a kind of critical slowing down. I undertook to estimate the optimal time for job submission and the critical exponent for slowing down at any finite order bifurcation point turned out to be $\Delta = 1$, the same universal "mean field" value in phase transition theory.[10] The derivation was so simple that I reproduce it here.

Let us consider a 1D map of the form $x_{n+1} = \mu f(x_\mu)$ and denote its pth iterate as

$$F(p, \mu, x) = \mu f(x) \circ \mu f(x) \circ \cdots \mu f(x) \circ \mu f(x).$$

The map $x_{n+1} = F(p, \mu, x_n)$ converges to a fixed point x^* near a pth order bifurcation at μ_p. In the vicinity of the fixed point let $x_n = x^* + \epsilon_n$. We have as usual

$$\epsilon_{n+1} = |F'(p, \mu, x^*)|\epsilon_n. \tag{1}$$

The convergence condition $|F'| < 1$ breaks down at the bifurcation parameter μ_p where

$$|F'(p, \mu_p, x^*)| = 1. \tag{2}$$

Assuming that ϵ_n diminishes as $e^{-n/\tau}$ where τ is a time constant governing the convergence rate, then from Eq. (1) it follows immediately that

$$\tau = -\left(\log |F'(p, \mu, x^*)|\right)^{-1} = -\left(p \log \mu + \sum_{i=1}^{p} \log f'(x_i^*)\right)^{-1}, \tag{3}$$

where x_i^* are the points forming the p-cycle. Expanding $\log \mu$ near μ_p, i.e., let

$$\log \mu = \log[\mu_p + (\mu - \mu_p)] = \log \mu_p - \frac{\mu - \mu_p}{\mu_p} + \cdots$$

and noticing

$$p \log \mu_p + \sum_{i=1}^{p} \log f'(x_i^*) = 0,$$

which follows from Eq. (2), we get

$$\tau = \frac{\mu_p}{p(\mu - \mu_p)}. \tag{4}$$

Comparing this result with the critical slowing down exponent Δ defined in the conventional theory of critical dynamics $\tau \propto |\mu - \mu_c|^{\Delta}$ we get the simple yet universal value $\Delta = 1$.

When I was happy improving the efficiency of job submission, Feigenbaum came, gave a talk and stayed for a couple of days. In our discussion Mitchell asked "why didn't you calculate the slowing down exponent near the infinite accumulation point of the period-doubling cascade?" Both of us did not realize that the answer was already there. The behavior of the Lyapunov exponent near the accumulation point μ_∞ has been shown to be[11,12]

$$\lambda(\mu_\infty - \mu) \propto (\mu_\infty - \mu)^t$$

and the exponent t is given by

$$t = \frac{\log 2}{\log \delta} = 0.44980\ldots,$$

where $\delta = 4.669\ldots$ is the first Feigenbaum constant. The slowing down exponent is nothing but the reciprocal of t, therefore $\Delta_\infty = 2.223\ldots$

From the discussion with Feigenbaum I realized that no one had calculated the critical slowing down exponent near a bifurcation point in one-dimensional maps. I wrote a one-page letter to *Physics Letters A* and it appeared without argument with the referees.[13] A few years later the same result $\Delta = 1$ was obtained by others[14] in a lengthy calculation.

We note that the above discussion concerns exponential slowing-down at μ approaching μ_p and eventuall μ_∞. Right at μ_∞ the exponential behavior changes to power-law as studied by Grassberger and Scheunert.[15]

4. OPERATIONAL DIMENSION OF TRANSIENT POINTS

Due to critical slowing down one can never get an isolated point in tracing the limiting set near the bifurcation parameter, no matter how close a fixed point is. There always appears an island of points converging to the fixed point. Let us derive the distribution of these points and measure the dimension of this ever shrinking island by applying the box-counting idea analytically.

For simplicity let us consider the first bifurcation parameter μ_1 in a period-doubling cascade of a unimodal map. Taking a parameter value μ very close to μ_1 and shifting the origin of the phase space to the fixed point, we can write the mapping as

$$x_{n+1} = -x_n + a x_n^2,$$

where a is determined by the second derivative or curvature of the map at the fixed point and x_n is a small deviation from the fixed point. The iterations fall at different sides of the fixed point alternatively. In order to study the distribution on one side of the fixed point we iterate once more and ignore higher powers of x_n to get

$$x_{n+2} = x_n - 2a^2 x_n^3.$$

When n is big enough this difference equation may be replaced by a differential equation:

$$\frac{dx}{dt} = -a^2 x^3. \tag{5}$$

If we introduce a distribution of the points $\rho(x)$ and write

$$dn = \rho(x) dx,$$

then it follows from Eq. (5) that

$$\rho(x) \propto \frac{1}{|x|^3}. \tag{6}$$

Now let us cover the distribution by using boxes of size ϵ. Since $\rho(x)$ drops down monotonically from infinity at $x = 0$ there always exists a x_0 such that

$$\epsilon \rho(x_0) = 1. \tag{7}$$

When $x < x_0$ there are points in every box, but when $x > x_0$ there is only one point in many adjacent boxes. Therefore, the total number of points is given by

$$N(\epsilon)=\frac{2x_0}{\epsilon}+2\int_{x_0}^{\infty}\rho(x)dx,$$

where the multiplier 2 takes into account contributions from both sides of the fixed point (it is an unimportant factor). Carrying out the integration and insert the x_0 obtained from Eq. (7), we get

$$N(\epsilon)\propto\epsilon^{-2/3}.$$

It follows from this relation that the island has a fractal dimension $D=2/3$.[16] We call this an "operational" dimension to distinguish it from the true dimension $D=0$.

By the way, the distribution of orbital points in an intermittency regime may be shown to be

$$\rho(x)\propto\frac{1}{|x|^2}, \tag{8}$$

which leads to a fractal dimension $D=1/2$.

5. "CROSS-OVER" OF DIMENSION

Now we have got a problem. When approaching any finite order period-doubling bifurcation point there is an "operational" dimension whose box-counting value is $D=2/3=0.666\ldots$ At the accumulation point of the period-doubling cascade the dimension ought to be $D_0=0.538\ldots$.[5] How do we reconcile these two distinct numbers? This reminds us about the crossover of some critical exponent in phase transitions. It turns out that there are two limits to be taken in a dimension calculation: the order p of the bifurcation which goes to infinity at the accumulation point and the box size ϵ which should vanish. It so happens that there is a function $D(p,\epsilon)$ and changing the order of taking the limits would lead to different results:[17]

$$\begin{aligned}\lim_{p\to\infty}\lim_{\epsilon\to 0}D(p,\epsilon)&=2/3=0.666\ldots,\\ \lim_{\epsilon\to 0}\lim_{p\to\infty}D(p,\epsilon)&=0.538\ldots\end{aligned} \tag{9}$$

The two parameters may be combined to form a single scaling parameter $\theta=\epsilon^{1/p}$ and the scaling function reads:

$$D(\theta)=\begin{cases}\frac{2}{3}+\frac{\gamma\ln\alpha}{\ln\theta}(D_0-\frac{2}{3}), & \theta>\alpha^{\gamma},\\ D_0, & \theta\leqslant\alpha^{\gamma},\end{cases}$$

where

$$\gamma = \frac{\ln 2}{D_0 \ln \alpha}$$

is a constant made of the universal constants D_0 and α.

By the way, the cross-over of the slowing down exponent from $\Delta = 1$ at finite bifurcation to 2.223 at the infinite accumulation might be treated in a similar way. No one has performed the analysis yet.

It is interesting to note that one can get most of the Grassberger value $D_0 = 0.538$ by using elementary scaling argument and simple arithmetic.

The scaling properties of one-dimensional maps are characterized by the Feigenbaum convergence rate δ in the parameter space, the scaling factor α in the phase space, the noise scaling factor κ, and the dimension D or even the D_q versus q spectrum of the limiting set. In fact, in a unimodal map one can pick up infinitely many period-n-tupling sequences besides the period-doubling ones.[18] All these sequences enjoy scaling properties similar to the period-doubling sequence. In particular, there are four series of "exponents" δ_W, α_W, κ_W, and D_q^W, where W is an admissible primitive symbolic word that determines the particular period-n-tupling sequence, the Feigenbaum exponents being the simplest case $W = R$. The δ_W and α_W were calculated in ref. 18, D_q^W – in ref. 19, and κ_W in ref. 20 for a number of primitive words W.

To be more precise, there are n different phase space scaling factors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ in a period-n-tupling sequence. In the period-doubling case the two factors are $\epsilon_1 = 1/\alpha$ and $\epsilon_2 = \epsilon_1^2$, where α is the Feigenbaum scaling factor α. Putting these ϵ_1 and ϵ_2 into the well-known sum rule (see, e.g., p. 350 in ref. 25)

$$\sum_{j=1}^{n} l_j^D = 1$$

for $n = 2$ we get a quadratic equation $y + y^2 = 1$ for $y = 1/\alpha^D$. Therefore, the dimension D is estimated to be

$$D \approx \frac{(\sqrt{5}+1)/2}{\log \alpha} = 0.524\ldots$$

It differs from the precise value in less than 2.6%. This result might be improved by taking into account the "corrections to scaling" as $\epsilon_2 = \alpha^{-2}$ is not an exact relation.

6. BOX-COUNTING RENORMALIZATION METHOD

The above calculation has made use of the "exact" scaling factor $\alpha = 2.5029$. Therefore, it cannot serve as a good start of a scaling theory. Inspired by the Suzuki cluster mean field renormalization theory Zheng developed a box-counting renormalization method to calculate the dimension of the limiting set at μ_∞ from studying finite periods.[21] The result[22] is summarized in Table I where nP stands for Period n and nI for n-Island chaotic band.

Table I. Dimension by Box-Counting Renormalization Method

	D_0	D_1	D_2
$1P \to 2P$	0.491		
$4I \to 8I$	0.5641		
$16I \to 32I$	0.5386	0.514	0.498
"Exact"	0.538045...[23]	0.517...[24]	0.500 ± 0.005[24]

For a period-n-tupling sequence with $n \geqslant 3$ there are n scaling factors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. No general relation among these numbers have been known except for $\epsilon_n = \epsilon_1^2$, which may be justified by renormalization group consideration (see, e.g., p. 357 in ref. 25). These ϵ_i have been obtained numerically and used to calculate the D_q and $f(\alpha)$ spectra for the limiting sets of various period-n-tupling sequences.[19]

7. IS THERE A SUPER-SCALING?

The D_q^W versus q as well as the $f_W(\alpha)$ versus α curves for different W, e.g., $W = R$,R, RLL, $RLLR$, etc., look quite similar to each other and there is a seemingly monotone dependence of the location of these curves to the order of these words in the symbolic dynamics of two letters.[19] Kefe Cao, a graduate student in the southernmost province Yunnan of China, succeeded in rescaling all these curves into one universal curve of D_q^W/D_0^W versus q/q_0 or $f_W(\alpha(q))/f_W(\alpha(0))$ versus $\alpha(q)/\alpha(0)$[26,27] (see Figs. 1–5 in ref. 29). These rescaled curves coincide within a precision of four digits that is good enough for naked eyes to accept as a single curve. As the universal curve no longer depends on the word W it might be called a "super-scaling" property as compared to the Feigenbaum scaling which depends on the word W. Such an interesting universal property should be simply provable.

The Yunnan University group published a few papers to prove the universal scaling.[28,29] The proof looked too sophisticated and it was hard to get the essence. Since it is a question of principle whether the "super-scaling" is an exact or an approximate property I made extensive numerical calculation trying to rescale the curves with precision higher than four digits. However, I could never go beyond four digits no matter what approach I took. Quite probably, the "super-scaling" is an approximate property which holds with about four digits of numerical precision. I consider this as an open problem.

It is a great pleasure to devote this paper to the 60th birthday of Feigenbaum.

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